Uitwerking Herkansing Analyse 8 Feb 2013

1. (a) Let $\epsilon > 0$. As f is uniformly continuous on A, there exists $\delta > 0$ such that $x, y \in A$ satisfying $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$. As (x_n) is cauchy, there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|x_n - x_m| < \delta$. Then, if $n, m \geq N$, we have $x_n, x_m \in A$ and $|x_n - x_m| < \delta$, so that $|f(x_n) - f(x_m)| < \epsilon$ by uniform continuity. This shows that $f(x_n)$ is cauchy. $\square$

   (b) No. Let $A = (0, \infty)$, $x_n = 1/n$ and $f(x) = 1/x$. Then $(x_n) \subseteq A$ is cauchy, f is continuous on A but $f(x_n) = n$ is not cauchy. $\square$

2. (a) Let x be a limit point of E. Then there exists a sequence (x_n) contained in $E \setminus \{x\}$ such that $x_n \to x$. Take any $\lambda \in \Lambda$. Then (x_n) is contained in E_λ as $E \subseteq E_\lambda$, so x is a limit point of E_λ, and as it is closed, it follows that $x \in E_\lambda$. But λ was arbitrary so $x \in E$, which shows that E is closed. $\square$

   (b) Let x be a limit point of E. Then there exists a sequence (x_n) contained in $E \setminus \{x\}$ such that $x_n \to x$. For each $n \in \mathbb{N}$, $x_n \in E_i$ for some $i \in \{1, 2, \ldots, m\}$. In fact, there exists an $i_0 \in \{1, 2, \ldots, m\}$ such that E_{i_0} containes an infinite number of elements of the sequence (x_n) (if not the sequence would be finite). Let (x_{n_k}) denote the subsequence of (x_n) obtained by deleting every term not in E_{i_0}. As there are infinitely many terms left this subsequence is well defined. Now, $x_n \to x$ as $n \to \infty$ implies $x_{n_k} \to x$ as $k \to \infty$, and we know that (x_{n_k}) is contained in $E_{i_0} \setminus \{x\}$, so x is a limit point for E_{i_0}, but it is closed so $x \in E_{i_0} \subseteq E$. This shows that E is closed. $\square$

   (c) No, $\bigcup_{n=1}^{\infty} [1/n, 1] = (0, 1]$. $\square$

3. (a) By uniform convergence (using the definition for $\epsilon = 1$ and $n = N$), there exists an $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < 1$ for all $x \in A$. f_N is bounded on A, so there exists $M > 0$ such that $|f_N(x)| \leq M$ for all $x \in A$. Then, $|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M$, which shows that f is bounded. $\square$

   (b) By uniform convergence, there exists an $N \in \mathbb{N}$ such that forall $n \geq N$ and $x \in A$ it follows that $|f_n(x) - f(x)| < 1$. As f is bounded, there exists $M' > 0$ such that $|f(x)| \leq M'$ for all $x \in A$. Then $|f_n(x)| = |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + M'$. Hence $|f_n(x)| < 1 + M'$ for all $n \geq N$ and $x \in A$, to get the remaining n we just use the boundedness of those and take the maximum of the bounds. For $n < N$, f_n is bounded so there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$. Let $M = \max\{M_1, M_2, \ldots, M_{N-1}, 1 + M'\}$. To show that this indeed works, let $n \in \mathbb{N}$ and $x \in A$ arbitrary. If $n \geq N$, $|f_n(x)| < 1 + M' \leq M$, if $n < N$, $|f_n(x)| \leq M_n \leq M$. This shows that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$. $\square$

4. (a) The pointwise limit f equals 0 if $|x| < 1$ and 1 if $x = \pm 1$. Each f_n is continuous on A while f is not, hence the convergence is not uniform. $\square$

(b) The pointwise limit f is the zero function, and the convergence is uniform. To see this, observe that $|f_n(x) - f(x)| = |\frac{\arctan(nx)}{n(x^2+1)}| \leq \frac{\pi}{2n}$. Let $\epsilon > 0$, as $\frac{\pi}{2n} \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\frac{\pi}{2n} < \epsilon$. Then, if $n \geq N$ and $x \in \mathbb{R}$, it follows that $|f_n(x) - f(x)| \leq \frac{\pi}{2n} < \epsilon$, which shows that the convergence is uniform on $\mathbb{R}$. $\square$

(c) The pointwise limit is the zero function, but the convergence is not uniform, to prove this we use the sequential criterion for non-uniform convergence. Let $n_k = k$ and $x_k = e^k - 1$. Then $n_k \to \infty$ as $k \to \infty$ and $(x_k) \subseteq A$. Then $|f_{n_k}(x_k) - f(x_k)| = \frac{1}{k}\log(e^k - 1 + 1) = \frac{k}{k} = 1$ for all $k \in \mathbb{N}$, so the convergence is not uniform on $[0, \infty)$. $\square$

5. (a) For any $x \in [0,1]$, we have $|x^n \sin(n\pi x)/n^2| \leq 1/n^2$ and $\sum 1/n^2$ converges, so the series converges uniformly on $[0,1]$ by the M-test. Each f_n is also continuous on $[0,1]$ so f is continuous on $[0,1]$ by uniform convergence.

(b) Let $x_0 \in [0,1)$. Pick $R \in \mathbb{R}$ such that $x_0 < R < 1$. Then for any $x \in [0,R]$, observe that

$$
\begin{aligned}
|f_n'(x)| &= \left|\frac{nx^{n-1}\sin(n\pi x) + n\pi x^n \cos(n\pi x)}{n^2}\right| = \frac{x^{n-1}}{n}|\sin(n\pi x) + \pi x \cos(n\pi x)| \\
&\leq \frac{R^{n-1}}{n}(|\sin(n\pi x) + \pi|x||\cos(n\pi x)|) \leq \frac{R^{n-1}}{n}(1 + \pi R)
\end{aligned}
$$

and $\sum \frac{R^{n-1}}{n}(1+\pi R)$ converges by comparison with the geometric series $\sum R^{n-1}(1+\pi R)$, so $\sum f_n'(x)$ converges uniformly on $[0,R]$, hence f is differentiable on $[0,R]$ so also at x_0. But $x_0 \in [0,1)$ was arbitrary so this shows that f is differentiable on $[0,1)$.

6. (a) Let P be a partition of $[0,1]$. Then for any subinterval $[x_{k-1}, x_k]$, there exists an $x \in [0,1]$ such that $f(x) = 0$, so $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = 0$, this shows that $L(f,P) = 0$.

(b) in the interval $[\epsilon/2, 1]$, f has a finite number of discontinuities, say M. These occur at $x = 1, 1/2, \ldots, 1/M$. Then consider

$$P_\epsilon = \{0, \frac{\epsilon}{2}, \frac{1}{M} - \frac{\epsilon}{4M}, \frac{1}{M} + \frac{\epsilon}{4M}, \ldots, \frac{1}{2} - \frac{\epsilon}{4M}, \frac{1}{2} + \frac{\epsilon}{4M}, 1 - \frac{\epsilon}{4M}, 1\}$$

For which it follows that $U(f, P_\epsilon) < \epsilon$.

(c) from (a) and (b) it follows that f is integrable on $[0,1]$, so $\int_0^1 f = L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} = \sup\{0 : P \in \mathcal{P}\} = 0$.